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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Thursday 13 November 2014 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 12]*

A group with the binary operation of multiplication modulo 15 is shown in the following Cayley table.

$\times_{15}$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	a	b	c
13	13	11	7	1	14	d	e	f
14	14	13	11	8	7	g	h	i

(a) Find the values represented by each of the letters in the table. *[3]*

(b) Find the order of each of the elements of the group. *[3]*

(c) Write down the three sets that form subgroups of order 2. *[2]*

(d) Find the three sets that form subgroups of order 4. *[4]*

2. *[Maximum mark: 8]*

Define $f : \mathbb{R} \setminus \{0.5\} \to \mathbb{R}$ by $f(x) = \dfrac{4x+1}{2x-1}$.

(a) Prove that f is an injection. *[4]*

(b) Prove that f is not a surjection. *[4]*

3. *[Maximum mark: 11]*

Consider the set A consisting of all the permutations of the integers $1, 2, 3, 4, 5$.

(a) Two members of A are given by $p = (1\,2\,5)$ and $q = (1\,3)(2\,5)$.
Find the single permutation which is equivalent to $q \circ p$. *[4]*

(b) State a permutation belonging to A of order

(i) 4;

(ii) 6. *[3]*

(c) Let $P = \{$all permutations in A where exactly two integers change position$\}$,

and $Q = \{$all permutations in A where the integer 1 changes position$\}$.

(i) List all the elements in $P \cap Q$.

(ii) Find $n(P \cap Q')$. *[4]*

4. *[Maximum mark: 10]*

The group $\{G, *\}$ has identity e_G and the group $\{H, \circ\}$ has identity e_H. A homomorphism f is such that $f: G \to H$. It is given that $f(e_G) = e_H$.

(a) Prove that for all $a \in G$, $f(a^{-1}) = \left(f(a)\right)^{-1}$. *[4]*

Let $\{H, \circ\}$ be the cyclic group of order seven, and let p be a generator.
Let $x \in G$ such that $f(x) = p^2$.

(b) Find $f(x^{-1})$. *[2]*

(c) Given that $f(x * y) = p$, find $f(y)$. *[4]*

5. *[Maximum mark: 19]*

(a) State Lagrange's theorem. *[2]*

$\{G, *\}$ is a group with identity element e. Let $a, b \in G$.

(b) Verify that the inverse of $a * b^{-1}$ is equal to $b * a^{-1}$. *[3]*

Let $\{H, *\}$ be a subgroup of $\{G, *\}$. Let R be a relation defined on G by

$$aRb \Leftrightarrow a * b^{-1} \in H.$$

(c) Prove that R is an equivalence relation, indicating clearly whenever you are using one of the four properties required of a group. *[8]*

(d) Show that $aRb \Leftrightarrow a \in Hb$, where Hb is the right coset of H containing b. *[3]*

It is given that the number of elements in any right coset of H is equal to the order of H.

(e) Explain how this fact together with parts (c) and (d) prove Lagrange's theorem. *[3]*